Student Number

Hunters Hill High School

2023 **TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/ or calculations

Total marks: 100

Section I – 10 marks (pages 3–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–13)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 Marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. Let z be a complex number such that $z^2 = \frac{i}{z}$.
Which of the following is a possible value for z?

(A) $0 + i$

(B) $-1 + i$

(C) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(D) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

2. Consider the statement: If all walls of this room are green, then this is a green room.
Which of the following statements is the contrapositive of this statement?

(A) If this is a green room, then all walls of this room are green

(B) If this is not a green room, then all walls of this room are not green

(C) If this is not a green room, then no walls of this room are green

(D) If this is a not green room, then there is at least one wall of this room which is not green

3. Simplify $\left(\frac{1}{2}\left(\sqrt{3}-i\right)\right)^{2023}$

(A) $\frac{1}{2}\left(-\sqrt{3}-i\right)$

(B) $\frac{1}{2}\left(-\sqrt{3}+i\right)$

(C) $\frac{1}{2}\left(\sqrt{3}-i\right)$

(D) $\frac{1}{2}\left(\sqrt{3}+i\right)$

4. Consider the vector $\underset{\sim}{u} = 3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$.

Which of the following vectors is perpendicular to $\underset{\sim}{u}$?

(A) $4\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$

(B) $3\underset{\sim}{i} - 1\underset{\sim}{j} + 5\underset{\sim}{k}$

(C) $2\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$

(D) $\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$

5. Which of the following is equivalent to the expression $\dfrac{3x+7}{(x+1)(x^2-2x-3)}$?

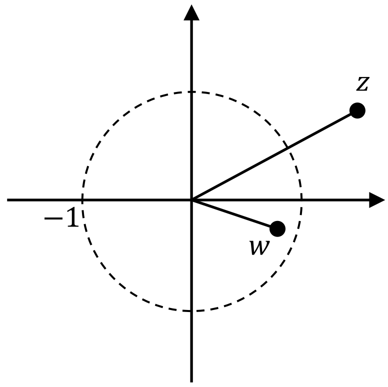
(A) $\dfrac{-1}{x+1}+\dfrac{1}{x-3}$

(B) $\dfrac{-1}{(x+1)^2}+\dfrac{1}{x-3}$

(C) $\dfrac{-1}{x+1}+\dfrac{x-1}{(x+1)^2}+\dfrac{1}{x-3}$

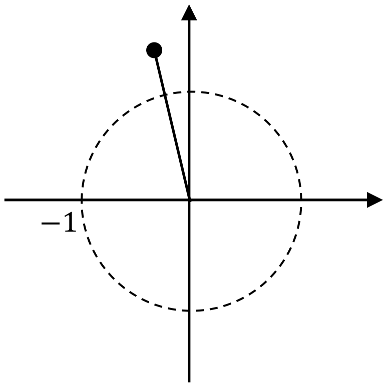
(D) $\dfrac{-1}{x+1}+\dfrac{-1}{(x+1)^2}+\dfrac{1}{x-3}$

6. The argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the fourth quadrant.

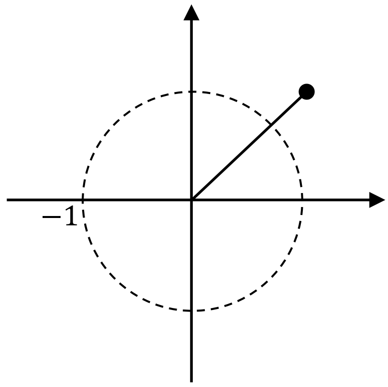


Which of the following diagrams best illustrates the complex number iwz?

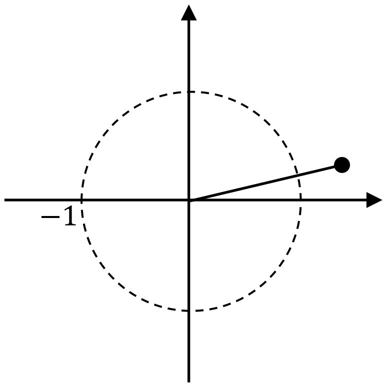
(A)



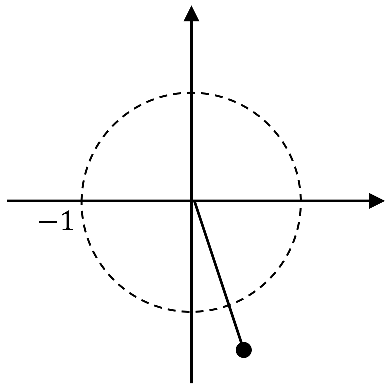
(B)



(C)



(D)



7. Which of the following expressions is equivalent to $\int -\frac{1}{\sqrt{1-3x^2}}\,dx$?

(A) $\frac{1}{6}\cos^{-1} 3x + c$

(B) $\frac{1}{6}\sin^{-1} 3x + c$

(C) $\frac{1}{\sqrt{3}}\cos^{-1} \sqrt{3}x + c$

(D) $\frac{1}{\sqrt{3}}\sin^{-1} \sqrt{3}x + c$

8. A particle is describing simple harmonic motion in a straight line with an amplitude of 6m. Its speed is 4m/s when the particle is 3m from the centre of motion.
What is the period of the motion?

(A) $\frac{3\sqrt{3}\pi}{2}$

(B) $\frac{9\pi}{2}$

(C) $\frac{3\sqrt{3}\pi}{4}$

(D) $\frac{9\pi}{4}$

9. Which of the following represents the vector projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ given $A(1,3,-2)$ and $B(2,-2,1)$?

(A) $\begin{bmatrix} -\frac{4}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{bmatrix}$

(B) $\begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$

(C) $\begin{bmatrix} -\frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{bmatrix}$

(D) $\begin{bmatrix} \frac{-2}{3} \\ \frac{-6}{3} \\ \frac{4}{3} \end{bmatrix}$

10. A particle is moving along a straight line so that initially its displacement is $x = 2$, its velocity is $v = 2$ and its acceleration is $a = 4$.
Which of the following is a possible equation of the motion of the particle?

(A) $v = x^2 + 2x$

(B) $v = 2 + 4\ln(x-1)$

(C) $v = 2\sin(x-2) + 2$

(D) $v^2 = 4(x^2 - 3)$

End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Begin each question in a new writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Begin a new Writing Booklet.

(a) Let $z = 1 - 4i$ and $w = 3 + 2i$.
Find:

(i) $\overline{w}$ **1**

(ii) $|wz|$ **2**

(iii) $\dfrac{\overline{w}}{z}$ **2**

(b) Find

(i) $\displaystyle\int \frac{dx}{\sqrt{5 + 2x - x^2}}$ **2**

(ii) $\displaystyle\int \frac{x^2 - 1}{x^2 + 4}\,dx$ **2**

(c) Solve $z^2 + (2 + i)z + 2 - 2i = 0$ **3**

(d) Prove by contradiction that $\sqrt{7}$ is irrational. **3**

End of Question 11

Question 12 (14 marks) Begin a new Writing Booklet.

(a) Using substitution of $x = \cos^2\theta$, evaluate **4**

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$$

(b) On an Argand diagram, sketch the region defined by **3**

$$-\frac{\pi}{4} < \arg(z - i) < \frac{\pi}{4}$$
$$Re(z) \leq 3$$

(c) Find $\displaystyle\int e^x \sin x\,dx$. **4**

(d) The point $P(x, y)$, representing the complex number z, moves in an Argand diagram such that $|z - 6| = |z + 2i|$.

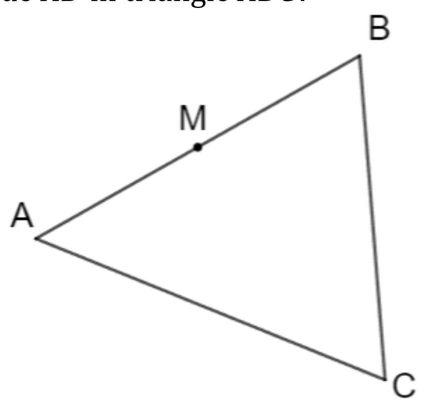
(i) Show that the path that P traces out has the formula $3x + y - 8 = 0$. **2**

(ii) Find the minimum value of $|z|$ as P moves along its path. **1**

End of Question 12

Question 13 (15 marks) Begin a new Writing Booklet.

(a) M is the midpoint of side AB in triangle ABC. **4**



Using vectors, show that $\left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{BC}\right|^2 = 2\left(\left|\overrightarrow{CM}\right|^2 + \left|\overrightarrow{AM}\right|^2\right)$.

(b) A sequence is defined recursively as $u_0 = 5, u_n = 2u_{n-1} + 1$ for $n \geq 1$. **3**

By induction, prove that $u_n = 3(2^{n+1}) - 1$ for all positive integers n.

(c) **(i)** Express $4i$ in exponential form. **1**

(ii) Hence, find the 4th roots of $4i$. **2**

(iii) Sketch the roots on an Argand diagram. **2**

(d) **(i)** Prove that $a^2 + b^2 \geq 2ab$ **1**

(ii) Hence, or otherwise, prove that $(p+2)(q+2)(p+q) \geq 16pq$, where p and q are positive real numbers. **2**

End of Question 13

Question 14 (15 marks) Begin a new Writing Booklet.

(a) A stone is released on the surface of the ocean, at which point it immediately sinks. Let gravity be $10\ \text{ms}^{-2}$ and the resistance due to the water is proportional to the square of the velocity.

(i) Explain why the acceleration of the stone can be given by **1**

$$a = 10 - \frac{k}{m}v^2.$$

(ii) Given $\frac{k}{m} = \frac{1}{40}$, show that $v = \frac{20(e^t - 1)}{e^t + 1}$. **4**

(iii) Using $a = v\frac{dv}{dx}$, show that $x = 20\ln\left(\frac{400}{400 - v^2}\right)$. **2**

(iv) How far does the stone sink in the first 3 seconds? **2**

(b) $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ are two lines with vector equations:

$$\underset{\sim}{r_1} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$
$$\underset{\sim}{r_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$$

where $\lambda, \mu \in \mathbb{R}$.

(i) Show that these two lines intersect. **2**

(ii) Find the angle between the lines. **1**

(iii) Find the shortest distance from the point $P(1, 2, 2)$ to the line **3**

$$\underset{\sim}{r_1} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

End of Question 14

Question 15 (16 marks) Begin a new Writing Booklet.

(a) Prove that $9^{n+1} - 2^{n+1}$ is divisible by 7 for all $n \geq 0$. **3**

(b) A particle moves in a straight line, experiencing simple harmonic motion. At time t seconds its displacement from a fixed point O in the line is x metres, given by:

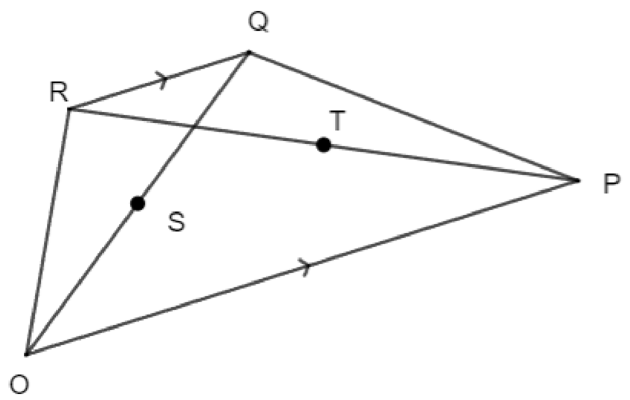
$$x = 1 + \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$$

(i) Show that $\ddot{x} = -(x-1)$ **1**

(ii) Find the time taken for the particle to first pass through the point O. **2**

(iii) Find in simplest exact form, the average speed of the particle during one complete oscillation of its motion. **2**

(c) $OPQR$ is a trapezium with $\overrightarrow{OP} = k\,\overrightarrow{RQ}$.
Let $\underset{\sim}{p} = \overrightarrow{OP}$ and $\underset{\sim}{r} = \overrightarrow{OR}$.
S and T are midpoints of OQ and PR respectively.



(i) Find $\overrightarrow{OS}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{r}$. **2**

(ii) Find $\overrightarrow{ST}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{r}$. **4**

(iii) Deduce the value of k required for $RSTQ$ to be a parallelogram. **2**

End of Question 15

Question 16 (15 marks) Begin a new Writing Booklet.

(a) Evaluate $\int \cos^3 x \sin^2 x \, dx$. **3**

(b) A stranded sailor, armed with a flare stands atop a headland, 40m above sea level. The flare is fired with a velocity of $20\underset{\sim}{i} - 6\underset{\sim}{j} + 12\underset{\sim}{k}$, where $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions, respectively. The acceleration of the cannonball due to the combined effects of gravity, air resistance and the wind is $-6\underset{\sim}{i} + 10\underset{\sim}{j} - 4\underset{\sim}{k}$.

(i) Find the displacement vector for the flare with respect to t. **3**

(ii) Find the highest point on the trajectory of the flare. **2**

(iii) Find the time taken for the flare to plunge into the ocean. **1**

(iv) At what angle to the horizontal was the flare fired? **1**

(c) Let $I_n = \int_0^1 (1 - x^2)^{\frac{n}{2}} dx$, where $n \geq 0$ is an integer.

(i) Show that **3**

$$I_n = \frac{n}{n+1} I_{n-2} \text{ for every integer } n \geq 2.$$

(ii) Hence evaluate I_5. **2**

End of Examination

2023 12MXX Trial – Solutions and Marking Guidelines

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SECTION I

1. $z^2 = \dfrac{i}{z}$

$z^3 = i$

$(0+i)^3 = -i$

$(-1+i)^3 = \left(\sqrt{2}\operatorname{cis}\frac{3\pi}{4}\right)^3 \neq i$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^3 = \left(\operatorname{cis}\frac{\pi}{6}\right)^3$

$= \operatorname{cis}\frac{\pi}{2} = i$

C.

2. $P \Rightarrow Q \longrightarrow \neg Q \Rightarrow \neg P$

for all $\rightarrow$ there exists

D

3. $\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{2023} = \left(\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^3\right)^{674}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$

$= i^{674}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$

$= i^{672}\cdot i^2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$

$= (+1)(-1)\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$

$= -\frac{\sqrt{3}}{2}+\frac{i}{2}$

i
-1
$-i$
$+1$

B

4. $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

A: $4(3)+4(-1)+2(-2) = 4$
B: $3(3)-1(-1)+5(-2) = 0$
C: $2(3)-3(-1)+3(-2) = 3$
D: $1(3)-5(-1)+3(-2) = 2$

B.

5. $\dfrac{3x+7}{(x+1)^2(x-3)} = \dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x-3}$

$3x+7 = A(x+1)(x-3)+B(x-3)+C(x+1)^2$

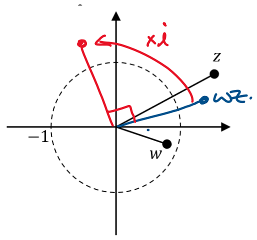
$x=-1 \Rightarrow 4 = -4B$
$B = -1$

$x = 3 \Rightarrow 16 = 4^2 C$
$C = 1$

$x^2: \Rightarrow 0 = A + C$
$A = -1$

D

6. product of $wz \Rightarrow$ add arguments
$\Rightarrow |w|<1$ so



$|wz| < |z|$

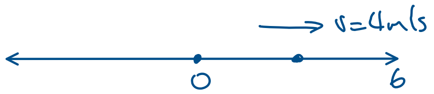
$\angle wz$ is a rotation $\frac{\pi}{2}$ anti-clockwise

A

7. $\frac{1}{\sqrt{3}}\int -\frac{\sqrt{3}\,dx}{\sqrt{1-(\sqrt{3}x)^2}} = \frac{1}{\sqrt{3}}\cos^{-1}\sqrt{3}x\,dx$

C

8.



$v^2 = n^2(a^2-x^2)$

$4^2 = n^2(6^2-3^2)$

$16 = n^2(36-9)$

$n^2 = \frac{16}{27}$

$n = \frac{4}{3\sqrt{3}} \Rightarrow P = \cancel{2}\pi \times \frac{3\sqrt{3}}{\cancel{4}_2}$

$= \frac{3\sqrt{3}\pi}{2}$

A

9.

$\frac{a \cdot b}{b \cdot b}\, b = \frac{1(2)+3(-2)-2(1)}{2^2+(-2)^2+1^2}\begin{bmatrix}2\\-2\\1\end{bmatrix}$

$= \frac{-6}{9}\begin{bmatrix}2\\-2\\1\end{bmatrix}$

$= -\frac{2}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix}$

A.

10. $x=2, v=2$ doesn't satisfy A or D

$v = 2\sin(x-2)+2$

$a = (2\sin(x-2)+2).2\cos(x-2)$ $\left(v\frac{dv}{dx}\right)$

$= (2\sin 0 + 2)\, 2\cos 0$

$= 4.$

C

Question 11

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a. $z = 1-4i \quad \omega = 3+2i$

i. $\bar{\omega} = 3-2i$

1- correct answer

ii. $|\omega z| = |(1-4i)(3+2i)|$
$= |11-10i|$
$= \sqrt{11^2+10^2}$
$= \sqrt{221}$

1- correct product of ωz or magnitude after error

2- correct answer

iii. $\dfrac{\bar{\omega}}{z} = \dfrac{3-2i}{1-4i} \times \dfrac{1+4i}{1+4i}$
$= \dfrac{3-2i+12i+8}{1+4^2}$
$= \dfrac{11+10i}{17}$

1- multiplies by conjugate

2- correct answer

b. i. $\displaystyle\int \frac{dx}{\sqrt{5+2x-x^2}} = \int \frac{dx}{\sqrt{6-(x^2-2x+1)}}$
$= \displaystyle\int \frac{dx}{\sqrt{6-(x-1)^2}}$
$= \sin^{-1}\dfrac{x-1}{\sqrt{6}} + c$

1- manipulates to find perfect square

2- correct ans. (ignore +c)

ii. $\displaystyle\int \frac{x^2-1}{x^2+4}\,dx = \int \frac{x^2+4-5}{x^2+4}\,dx$
$= \displaystyle\int \left(1 - \frac{5}{x^2+4}\right)dx$
$= x - \dfrac{5}{2}\tan^{-1}\dfrac{x}{2} + c$

1- correctly separates fraction

2- correct ans (ignore c)

c. $z^2+(2+i)z+2-2i=0$

$z = \dfrac{-(2+i) \pm \sqrt{(2+i)^2-4(1)(2-2i)}}{2(1)}$

1- correct use of quadratic formula

$\Delta = 3+4i-8+8i$
$= -5+12i$

let $(x+iy)^2 = -5+12i$

Re+Im $x^2-y^2=-5$
$2xy = 12 \Rightarrow xy = 6$

2- solution for x and y.

by observation $x=2, y=3$

$\therefore \sqrt{-5+12i} = \pm(2+3i)$

$\therefore z = \dfrac{-2-i \pm (2+3i)}{2}$
$= \dfrac{2i}{2}, \dfrac{-4-4i}{2}$
$= i, -2-2i$

3- correct ans

d. let $\sqrt{7}$ be rational

$\therefore \sqrt{7} = \frac{a}{b}$ $\quad a, b \in \mathbb{R}$

and a and b have no common divisor other than 1.

$a = b\sqrt{7}$

$a^2 = 7b^2$

as 7 divides a^2, then 7 must divide a.

hence $a = 7c$ for constant c.

$\therefore (7c)^2 = 7b^2$

and $7c^2 = b^2$

as 7 divides b^2, then 7 must divide b.

$\therefore$ both a and b have a factor of 7, contradicting the earlier assumption

$\therefore$ by contradiction, $\sqrt{7}$ is irrational

1 - assumes $\sqrt{7}$ is rational in form $\frac{a}{b}$

2 - deduces that 7 divides a, or equivalent.

3 - complete proof.

Question 12

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a) $x = \cos^2\theta$

$dx = 2\cos\theta.(-\sin\theta)d\theta$ — 1- correct dx

for $x = 0$, $\theta = \frac{\pi}{2}$

$x = \frac{1}{2}$, $\theta = \frac{\pi}{4}$

$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx = -2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{\frac{\cos^2}{1-\cos^2\theta}} \cdot \cos\theta\sin\theta\,d\theta$ — 2- correct substitution, including bounds

$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{\cos^2\theta}{\sin^2\theta}} \cdot \cos\theta\sin\theta\,d\theta$

$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\cancel{\sin\theta}} \cdot \cos\theta\cancel{\sin\theta}\,d\theta$

$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \cos^2\theta\,d\theta$ — 3- correct simplification

$\cos 2x = 2\cos^2\theta - 1$

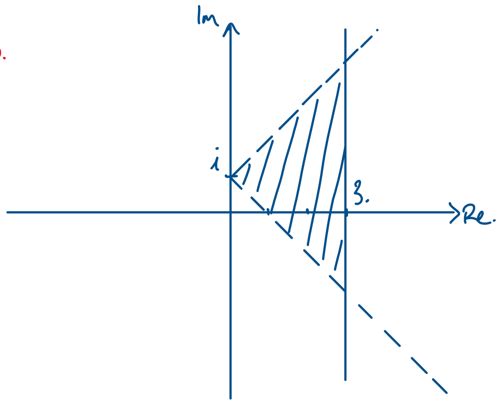
$= \frac{\cancel{2}}{\cancel{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta$

$= \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right)$

$= \frac{\pi}{4} - \frac{1}{2}$ — 4 - ans

b.



1- any correct line

$\arg(z-i) = \pm\frac{\pi}{4}$

$Re(z) = 3$

2- a correct region

3- ans

c. $\int e^x \sin x\,dx$

let $u = \sin x$ $\quad dv = e^x dx.$ — 1-correct du and v.

$du = \cos x\,dx$ $\quad v = e^x$

$\int e^x \sin x\,dx = e^x\sin x - \int e^x\cos x\,dx$ — 2-correct use of by parts

let $u = \cos x$ $\quad dv = e^x dx$

$du = -\sin x\,dx$ $\quad v = e^x$

$= e^x\sin x - \left(e^x\cos x - \int e^x(-\sin x)dx\right)$ — 3- second use of by parts

$= e^x(\sin x - \cos x) - \int e^x\sin x\,dx$

$$= e^x(\sin x - \cos x) - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\therefore \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$$

4 – ans

d. i. $|z-6| = |z+2i|$

$$|x+iy-6| = |x+iy+2i|$$
$$|x-6+iy|^2 = |x+i(y+2)|^2$$

1 – correct magnitude expression for one side

$$(x-6)^2 + y^2 = x^2 + (y+2)^2.$$
$$\cancel{x^2} - 12x + 36 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} + 4y + 4$$

$$12x + 4y - 32 = 0$$
$$\therefore 3x + y - 8 = 0$$

2 – ans

ii. Minimum value occurs when z is nearest the origin $\Rightarrow$ distance from $(0,0)$ to $3x+y-8=0$

$$d = \frac{|3(0)+0-8|}{\sqrt{3^2+1^2}}$$
$$= \frac{|-8|}{\sqrt{10}}$$
$$= \frac{4\sqrt{10}}{5}$$

1 – correct ans.

Question 13

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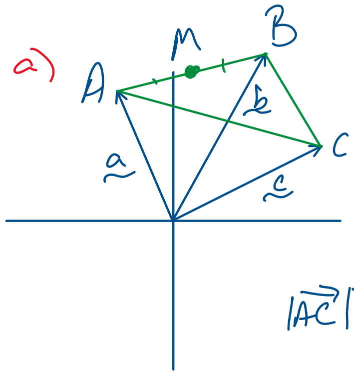
a)



$$\overrightarrow{OM} = \underset{\sim}{a} + \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

$$|\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2 = (\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{c}-\underset{\sim}{a}) + (\underset{\sim}{c}-\underset{\sim}{b})\cdot(\underset{\sim}{c}-\underset{\sim}{b})$$
$$= \underset{\sim}{c}^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}^2 + \underset{\sim}{c}^2 - 2\underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}^2$$

$$|\overrightarrow{CM}|^2 + |\overrightarrow{AM}|^2 = \left(\tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b}) - \underset{\sim}{c}\right)\cdot\left(\tfrac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b}) - \underset{\sim}{c}\right) + \left(\tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b}) - \underset{\sim}{a}\right)\cdot\left(\tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b}) - \underset{\sim}{a}\right)$$
$$= \tfrac{1}{4}(\underset{\sim}{a}+\underset{\sim}{b})^2 - (\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c} + \underset{\sim}{c}^2 + \tfrac{1}{4}(\underset{\sim}{a}+\underset{\sim}{b})^2 - \underset{\sim}{a}\cdot(\underset{\sim}{a}+\underset{\sim}{b}) + \underset{\sim}{a}^2$$
$$= \tfrac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})^2 - \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{c}^2 - \cancel{\underset{\sim}{a}\cdot\underset{\sim}{a}} - \underset{\sim}{a}\cdot\underset{\sim}{b} + \cancel{\underset{\sim}{a}^2}$$
$$= \tfrac{1}{2}\underset{\sim}{a}^2 + \cancel{\underset{\sim}{a}\cdot\underset{\sim}{b}} + \tfrac{1}{2}\underset{\sim}{b}^2 - \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{c}^2 - \cancel{\underset{\sim}{a}\cdot\underset{\sim}{b}}$$
$$= \tfrac{1}{2}\left(\underset{\sim}{a}^2 + \underset{\sim}{b}^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{c} - 2\underset{\sim}{b}\cdot\underset{\sim}{c} + 2\underset{\sim}{c}^2\right)$$

$$\therefore\ |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2 = 2\left(|\overrightarrow{CM}|^2 + |\overrightarrow{AM}|^2\right)$$

b) $u_0 = 5,\ u_n = 2u_{n-1} + 1 \quad n \geq 1$

Prove true for $n=0$

$$u_0 = 3(2^{0+1}) - 1$$
$$= 3 \times 2 - 1$$
$$= 5$$

$\therefore$ expression true for $n=0$

Assume true for $n=k$

$\therefore\ u_k = 3(2^{k+1}) - 1$ – induction hypothesis

Prove true for $n=k+1$

i.e. $u_{k+1} = 3(2^{k+2}) - 1$

$$u_{k+1} = 2u_k + 1$$
$$= 2\times(3(2^{k+1}) - 1) + 1 \quad \text{– by induction hypothesis}$$
$$= 3(2 \times 2^{k+1}) - 2 + 1$$
$$= 3(2^{k+2}) - 1 \quad \text{as required}$$

$\therefore$ by the process of induction, expression's true for all positive integers n.

1 – finds

2 – expands
$|\overrightarrow{PR}$
or eq. w

3 – approp.
with bot

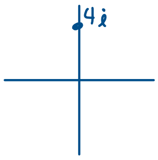
4 – correct s

1 – proves bas

2 – correctly
hypothes

3 – correct pro

c. i. $4i = 4e^{\frac{i\pi}{2}}$



1-correct as

ii. let $z^4 = 4i$

$= 4e^{i\left(\frac{\pi}{2}+2\pi k\right)}$

$\therefore z = \sqrt{2}\, e^{\frac{\pi i}{8}(1+4k)}$

1- correct form

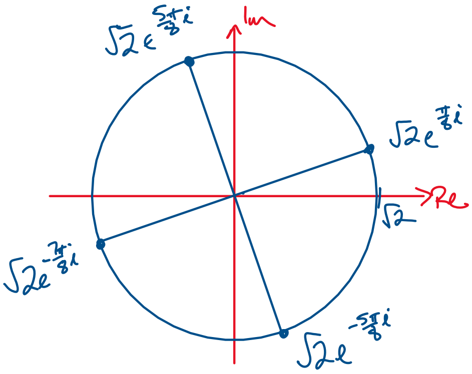
$\therefore z = \sqrt{2}e^{\frac{i\pi}{8}}, \sqrt{2}e^{i\frac{5\pi}{8}}, \sqrt{2}e^{i\frac{9\pi}{8}}, \sqrt{2}e^{i\frac{13\pi}{8}}$

Correct this to -3/8 instead of -5/8

$= \sqrt{2}e^{\frac{i\pi}{8}}, \sqrt{2}e^{i\frac{5\pi}{8}}, \sqrt{2}e^{-\frac{7\pi}{8}i}, \sqrt{2}e^{-\frac{5\pi}{8}i}$

2-correct r

iii.



1- 4 equally sp
on circle
OR

2- correct sk

d. i. $(a-b)^2 \geq 0$ for all $a,b>0$.

$a^2-2ab+b^2 \geq 0$

$\therefore a^2+b^2 \geq 2ab$

1-correct pr

ii. from i)

$(\sqrt{p})^2 + (\sqrt{2})^2 \geq 2\sqrt{p}.\sqrt{2}$

$(\sqrt{q})^2 + (\sqrt{2})^2 \geq 2\sqrt{q}.\sqrt{2}$

$(\sqrt{p})^2 + (\sqrt{q})^2 \geq 2\sqrt{p}.\sqrt{q}$

1-notes one a
these

$\therefore (p+2)(q+2)(p+q) = 2\sqrt{2}\sqrt{p} . 2\sqrt{2}.\sqrt{q} . 2\sqrt{p}\sqrt{q}$.

$= 8pq$.

2- correct pr

Question 14

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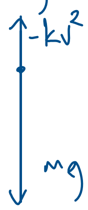
a) i. Taking downwards as positive direction with resistance acting against it.



So $ma = mg - kv^2$

and $a = g - \frac{k}{m}v^2$

1 - reasonable explanation noting direction of travel

ii. $\frac{k}{m} = \frac{1}{40}$

$$\frac{dv}{dt} = 10 - \frac{1}{40}v^2$$
$$= \frac{1}{40}(400 - v^2)$$
$$\int_0^v \frac{dv}{400-v^2} = \int_0^t \frac{1}{40}\,dt$$

1 - correct separation

let $\frac{1}{400-v^2} = \frac{A}{20-v} + \frac{B}{20+v}$

$1 = A(20+v) + B(20-v)$

$\Rightarrow A = B = \frac{1}{40}$

2 - partial fractions used correctly

$$\therefore \frac{1}{40}\int_0^v \left(\frac{1}{20-v} + \frac{1}{20+v}\right)dv = \frac{t}{40}$$
$$\Big[-\ln|20-v| + \ln|20+v|\Big]_0^v = t$$
$$\ln\left(\frac{20+v}{20-v}\right) - \ln\left(\frac{20}{20}\right) = t$$
$$\therefore \ln\left(\frac{20+v}{20-v}\right) = t$$

3 - correct integration

$$\frac{20+v}{20-v} = e^t$$
$$20+v = e^t(20-v)$$
$$v(1+e^t) = 20(e^t-1)$$
$$\therefore v = 20\left(\frac{e^t-1}{e^t+1}\right)$$

4 - ans shown

iii. $v\frac{dv}{dx} = \frac{1}{40}(400-v^2)$

$$\frac{1}{-2}\int_0^v \frac{-2v\,dv}{400-v^2} = \frac{1}{40}\int_0^x dx.$$

1 - correct separation

$$-\frac{1}{2}\Big[\ln|400-v^2|\Big]_0^v = \frac{x}{40}$$
$$x = -20\left(\ln|400-v^2| - \ln(400)\right)$$
$$= 20\left(\ln 400 - \ln(400-v^2)\right)$$
$$= 20\ln\left(\frac{400}{400-v^2}\right)$$

2 - ans shown

iv $\quad at = 3 \qquad v = \dfrac{20(e^3-1)}{e^3+1}$

$= 18.10296507 \text{ ms}^{-1}$ — 1- correct v

$x = 20\ln\left(\dfrac{400}{400-18.103^2}\right)$ — 2-correct x

$= 34.21760684 \text{ m}$

b) i. $\underset{\sim}{r}_1 = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\underset{\sim}{r}_2 = \begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\-4\end{pmatrix}$

intersect when $2 + 0\lambda = 1 + \mu$ - i

$0 - \lambda = 2 + 3\mu$ - ii

$1 + \lambda = 0 - 4\mu$ - iii

from i $\mu = 1$ — 1- finds correct μ

from ii $-\lambda = 2 + 3(1)$

$\lambda = -5$

given iii

LHS $= 1 - 5 = -4$ $\quad$ RHS $= -4(1) = -4 = $ LHS

$\therefore$ the lines intersect. — 2-confirms intersection

ii. let direction vectors $\underset{\sim}{d}_1 = (0,-1,1)$ and $\underset{\sim}{d}_2 = (1,3,-4)$

$\cos\theta = \dfrac{\underset{\sim}{d}_1 \cdot \underset{\sim}{d}_2}{|\underset{\sim}{d}_1|\,|\underset{\sim}{d}_2|}$

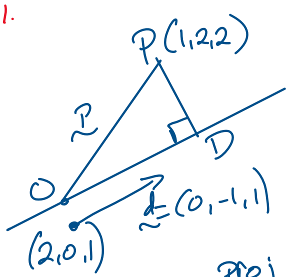
$= \dfrac{0(1) - 1(3) + -4}{\sqrt{0^2+(-1)^2+1^2}\sqrt{1^2+3^2+(-4)^2}}$

$= \dfrac{-7}{\sqrt{2}\cdot\sqrt{26}}$

$= \dfrac{-7}{2\sqrt{13}}$

$\theta = 166.1°$ — 1- correct angle.

iii.



$\underset{\sim}{p} = (-1, 2, 1)$

$\text{proj}_{\underset{\sim}{d}}\underset{\sim}{p} = \dfrac{(0,-1,1)\cdot(-1,2,1)}{(0,-1,1)\cdot(0,-1,1)}\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$= \dfrac{-2+1}{1+1}\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix}$ — 1- finds projection

$$DP = \underset{\sim}{p} - \text{proj}_{\underset{\sim}{d}} \underset{\sim}{p}$$
$$= (-1,2,1) - \left(0, \frac{1}{2}, -\frac{1}{2}\right)$$
$$= \left(-1, \frac{3}{2}, \frac{3}{2}\right)$$

2- finds perpendicular vector

$$|\overrightarrow{DP}| = \sqrt{(-1)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$
$$= \sqrt{\frac{11}{2}}$$
$$= \frac{\sqrt{22}}{2} \text{ units}$$

3- finds distance.

Question 15

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a. Prove base case, $n=0$

$$9^{0+1}-2^{0+1}=9-2$$
$$=7$$

which is divisible by 7, hence true for $n=0$.

Assume true for $n=k$

$\therefore 9^{k+1}-2^{k+1}=7M$, $M\in\mathbb{Z}$.

and $9^{k+1}=7M+2^{k+1}$ – induction hypothesis

Prove true for $n=k+1$

$$9^{(k+1)+1}-2^{(k+1)+1}=9.9^{k+1}+2.2^{k+1}$$
$$=9(7M-2^{k+1})+2.2^{k+1}$$ by induction hypothesis
$$=7(9M)-9.2^{k+1}+2.2^{k+1}$$
$$=7(9M)-7.2^{k+1}$$
$$=7\left[9M-2^{k+1}\right]$$

which is divisible by 7 as $9M-2^{k+1}\in\mathbb{Z}$.

$\therefore$ by principle of mathematical induction, expression divisible by 7 for $n\geq 0$.

b. i. $x=1+\sqrt{2}\cos\left(t-\frac{\pi}{4}\right) \Rightarrow \sqrt{2}\cos\left(t-\frac{\pi}{4}\right)=x-1$

$$\dot{x}=-\sqrt{2}\sin\left(t-\frac{\pi}{4}\right)$$
$$\ddot{x}=-\sqrt{2}\cos\left(t-\frac{\pi}{4}\right)$$
$$=-(x-1)$$

ii. Solve, $1+\sqrt{2}\cos\left(t-\frac{\pi}{4}\right)=0$

$$\cos\left(t-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$
$$t-\frac{\pi}{4}=(2n+1)\pi\pm\frac{\pi}{4}\quad n\in\mathbb{Z}.$$
$$t=(2n+1)\pi,\ (2n+1)\pi+\frac{\pi}{2}$$
$$=\pi,\ \frac{3\pi}{2},\ 3\pi,\ \frac{7\pi}{2},\ \ldots$$

$\therefore$ Particle first passes through 0 at π seconds

iii. distance travelled during one oscilation is $4\times$ amplitude

so $d=4\sqrt{2}$

1 – proves base case true,

2 – correct use of hypothesis
– correct induction step. w/ incorrect base case

3 – correct proof

1 – correctly shown

1 – shows expression

2 – correct ans.

1 – finds distance

so $d = 4\sqrt{2}$ — times distance travelled

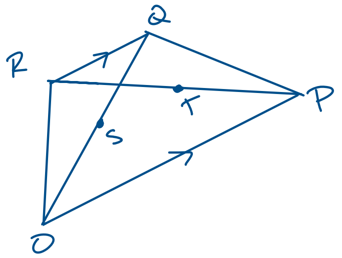
period $= 2\pi$

so average speed $= \dfrac{4\sqrt{2}}{2\pi}$

$= \dfrac{2\sqrt{2}}{\pi}$

$= 0.900316 \text{ ms}^{-1}$ — 2- correct ans

C. i.



$\vec{OS} = \frac{1}{2}(\vec{OQ})$

$= \frac{1}{2}(\vec{OR} + \vec{RQ})$ — 1- $\vec{OS}$ in terms of $\vec{OR}$ and $\vec{RQ}$

$= \frac{1}{2}(\vec{OR} + \frac{1}{k}\vec{OP})$

$= \frac{1}{2}(\underset{\sim}{r} + \frac{1}{k}\underset{\sim}{p})$ — 2- ans

ii. $\vec{OT} = \vec{OR} + \frac{1}{2}\vec{RP}$ — 1- in terms of $\vec{OR}$ and $\vec{RP}$

$= \vec{OR} + \frac{1}{2}(\vec{OP} - \vec{OR})$

$= \frac{1}{2}(\underset{\sim}{r} + \underset{\sim}{p})$ — 2- in terms of $\underset{\sim}{r}$ and $\underset{\sim}{p}$

$\vec{ST} = \vec{OT} - \vec{OS}$ — 3- correct difference for $\vec{ST}$

$= \frac{1}{2}(\underset{\sim}{r} + \underset{\sim}{p}) - \frac{1}{2}(\underset{\sim}{r} + \frac{1}{k}\underset{\sim}{p})$

$= \frac{1}{2}\underset{\sim}{p}(1 - \frac{1}{k})$

$= \frac{k-1}{2k}\underset{\sim}{p}$ — 4- ans

iii. for RSTQ to be a parallelogram,

$\vec{RQ} = \vec{ST}$ — 1- equates vectors

$\frac{1}{k}\underset{\sim}{p} = \frac{k-1}{2k}\underset{\sim}{p}$

$\therefore \; 2 = k - 1$

$k = 3$ — 2- correct k.

Question 16

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a)

$$\int \cos^3 x \sin^2 x \, dx$$
$$= \int (1-\sin^2 x)\sin^2 x \cdot \cos x \, dx$$
$$= \int (\sin^2 x - \sin^4 x) \cos x \, dx$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{3} + c$$

1- splits off cosx
2- expands si
3- integrates

b) i.

$$\ddot{x} = -6\underset{\sim}{i} + 10\underset{\sim}{j} - 4\underset{\sim}{k}$$
$$\dot{x} = -6t\underset{\sim}{i} + 10t\underset{\sim}{j} - 4t\underset{\sim}{k} + c$$

1- integrates correctly

as $\dot{x} = 20\underset{\sim}{i} - 6\underset{\sim}{j} + 12\underset{\sim}{k}$ at $t=0$

then $c = 20\underset{\sim}{i} - 6\underset{\sim}{j} + 12\underset{\sim}{k}$

2- find c.

so $\dot{x} = (20-6t)\underset{\sim}{i} + (10t-6)\underset{\sim}{j} + (12-4t)\underset{\sim}{k}$

$$x = (20t-3t^2)\underset{\sim}{i} + (5t^2-6t)\underset{\sim}{j} + (12t-2t^2)\underset{\sim}{k} + c$$

at $t=0$, $c = x = 40\underset{\sim}{k}$

So $x = (20t-3t^2)\underset{\sim}{i} + (5t^2-6t)\underset{\sim}{j} + (12t-2t^2+40)\underset{\sim}{k}$

3-correct ans

ii highest point when the vertical component of $\dot{x}$ is 0

i.e. $12-4t = 0$

$t = 3$

$$x = (20(3)-3(3)^2)\underset{\sim}{i} + (5(3)^2-6(3))\underset{\sim}{j} + (12(3)-2(3)^2+40)\underset{\sim}{k}$$
$$= 33\underset{\sim}{i} + 27\underset{\sim}{j} + 58\underset{\sim}{k}$$

Highest point is $(33, 27, 58)$

1- correct height 58 m
2- correct ans

iii flare reaches ocean when vertical component of x is 0

$\therefore 40 + 12t - 2t^2 = 0$

$20 + 6t - t^2 = 0$

$20 + 9 - (t^2 - 6t + 9) = 0$

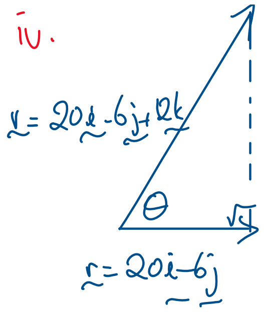
$(t-3)^2 = 29$

$t = 3 + \sqrt{29}$ disregard $t<0$

$= 8.385\,1648$ s

1 - correct ans

iv.



$$\cos\theta = \frac{\underset{\sim}{v}\cdot\underset{\sim}{r}}{|\underset{\sim}{v}||\underset{\sim}{r}|}$$

$$= \frac{20^2 + (-6)^2 + 12(0)}{\sqrt{20^2+(-6)^2+12^2}\ \sqrt{20^2+(-6)^2}}$$

$$= \sqrt{\frac{436}{580}}$$

$$= 0.86702026$$

$$\theta = 29.88580^\circ$$

1 - correct ans

c) $I_n = \int_0^1 (1-x^2)^{\frac{n}{2}}\,dx$

$u = (1-x^2)^{\frac{n}{2}}$ $\qquad dv = dx$

$du = \frac{n}{2}(-2x)(1-x^2)^{\frac{n-2}{2}}$ $\qquad v = x$

1 - correct parts

$$I_n = \left[x(1-x^2)^{\frac{n}{2}}\right]_0^1 - \frac{2n}{2}\int_0^1 -x^2(1-x^2)^{\frac{n-2}{2}}\,dx$$

2 - correct subts.

$$= -n\int_0^1 -x^2(1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= -n\int_0^1 \left((1-x^2)^{\frac{n-2}{2}} - x^2(1-x^2)^{\frac{n-2}{2}} - (1-x^2)^{\frac{n-2}{2}}\right)dx$$

$$= -n\int_0^1 (1-x^2)(1-x^2)^{\frac{n-2}{2}}\,dx + n\int_0^1 (1-x^2)^{\frac{n-2}{2}}\,dx$$

$\not\!\!\alpha$ - appropriate manipulation

$$= -n\int_0^1 (1-x^2)^{\frac{n}{2}}\,dx + n\int_0^1 (1-x^2)^{\frac{n-2}{2}}\,dx$$

$$= -nI_n + nI_{n-2}$$

$$\therefore (n+1)I_n = nI_{n-2}$$

$$I_n = \frac{n}{n+1}I_{n-2}$$

3 - correctly shown

iii $I_1 = \int_0^1 (1-x^2)^{\frac{1}{2}}\,dx$ $\qquad$ let $x = \cos\theta$ $\qquad$ $x=1,\ \theta=0$

$dx = -\sin\theta\, d\theta$ $\qquad$ $x=0,\ \theta=\frac{\pi}{2}$

$$= -\int_{\frac{\pi}{2}}^0 \sqrt{1-\cos^2\theta}\cdot\sin\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta \qquad \cos 2\theta = 1 - 2\sin^2$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{2}(0) - \left(0 + \frac{1}{2}(0)\right)\right)$$

$$= \frac{\pi}{4}$$

1 - value for I_1

$$I_3 = \frac{3}{3+1}I_1$$

$$= \frac{3}{4}\cdot\frac{\pi}{4}$$

$= \frac{3\pi}{16}$

$I_5 = \frac{5}{5+1} \cdot I_3$

$= \frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}^1\pi}{16}$

$= \frac{5\pi}{32}.$

2 - ans